Time-frequency Techniques in Biomedical Signal Analysis*

A Tutorial Review of Similarities and Differences

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Summary
Objectives: This review outlines the methodological fundamentals of the most frequently used non-parametric time-frequency analysis techniques in biomedicine and their main properties, as well as providing decision aids concerning their applications.

Methods: The short-term Fourier transform (STFT), the Gabor transform (GT), the S-transform (ST), the continuous Morlet wavelet transform (CMWT), and the Hilbert transform (HT) are introduced as linear transforms by using a unified concept of the time-frequency representation which is based on a standardized analytic signal. The Wigner-Ville distribution (WVD) serves as an example of the ‘quadratic transforms’ class. The combination of WVD and GT with the matching pursuit (MP) decomposition and that of the HT with the empirical mode decomposition (EMD) are explained; these belong to the class of signal-adaptive approaches.

Results: Similarities between linear transforms are demonstrated and differences with regard to the time-frequency resolution and interference (cross) terms are presented in detail. By means of simulated signals the effects of different time-frequency resolutions of the GT, CMWT, and WVD as well as the resolution-related properties of the interference (cross) terms are shown. The method-inherent drawbacks and their consequences for the application of the time-frequency techniques are demonstrated by instantaneous amplitude, frequency and phase measures and related time-frequency representations (spectrogram, scalogram, time-frequency distribution, phase-locking maps) of measured magnetoencephalographic (MEG) signals.

Conclusions: The appropriate selection of a method and its parameter settings will ensure readability of the time-frequency representations and reliability of results. When the time-frequency characteristics of a signal strongly correspond with the time-frequency resolution of the analysis then a method may be considered ‘optimal’. The MP-based signal-adaptive approaches are preferred as these provide an appropriate time-frequency resolution for all frequencies while simultaneously reducing interference (cross) terms.

1. Introduction

Biomedical signals are characterized by time-varying signal properties, i.e. from the statistical perspective, they are non-stationary. Therefore, time-variant signal processing methods are part of the standard repertoire in biomedical signal analysis. Both the frequency of occurrence as well as the shape and time-frequency characteristics of transient signal components have a high diagnostic value [1] leading to more frequently applications of time-variant analysis methods. Transient, spindle-like oscillations are almost always synonymous with transient signal components in EEG/MEG analysis. Therefore, brain computer interface and epileptic seizure prediction research are inconceivable without the use of time-frequency processing techniques. Another example is the time-variant analysis of the cardiorespiratory system, where the main goal is the quantification of short-term and long-term variations of ongoing oscillations (respiratory movements, blood pressure waves). The surface electromyogram (EMG) is an integrated part of gait analyses in which all measures are time-variant. There could be many more examples like these.

Signal components with time-variant properties may occur at different frequencies. Therefore, time-variant, frequency-selective approaches are required for their “time-frequency” analysis. The algorithms which make time-frequency analysis possible are called time-frequency techniques. The best-known and most frequently applied techniques in biomedicine are the short-time Fourier transform (STFT) or Gabor transform (GT), the continuous
Morlet wavelet transform (CMWT), and the Hilbert transform (HT). In the last decade, HT has had a renaissance with an increasing number of investigations regarding the role of synchronization of neuronal discharges in the brain [2]. It was A. Bruns who posed the question “Fourier-, Hilbert- and wavelet-based signal analysis: are they really different approaches?” [3]. The underlying mathematics tells us that there are differences, but they can be equivalent if a certain window function (Fourier-), a certain filter (Hilbert-), and a certain kernel envelope (wavelet-transform) are used [4]. However, the method’s users, then and today, were and are uncertain by “messages” with regard to the most powerful and/or appropriate method which have been originated by user and/or developer communities. In view of the significant progress in the advancement of time-frequency methods during the last decade this question must be re-addressed by including most advanced approaches. Due to the variety of the time-frequency approaches we focus this comparative overview on non-parametric, linear time-frequency techniques: STFT, GT, S-transform (ST), CMWT, HT as well as the matched Gabor transform (MGT) and the Hilbert-Huang transform (HHT) as signal-adaptive extensions. From the class of quadratic transforms (quadratic time-frequency distributions) the most frequently used Wigner-Ville distribution (WVD) and its combination with the matching pursuit (MP) decomposition will be integrated. An excellent overview of quadratic transforms and their relationship to linear transforms is given by B. Boashash [5] as well as Hlawatsch and Boudreaux-Bartels [6]. Time-variant parametric approaches which are based on autoregressive (AR) and autoregressive moving average (ARMA) models are not included here. Time-variant, multivariate AR models are essential in dynamic brain connectivity analysis. An overview of these types of approaches was published by our group in this journal [1].

This review article is organized as follows: Section 2 will provide a brief overview on the MEG data that were used. In Section 3.1, we will introduce the concept of the (complex) analytic signal for a cosine signal. This concept will be stepwise generalized towards amplitude-modulated and frequency modulated mono-components signals whose superposition forms a specific class of multi-component signals. The above mentioned linear transforms can be seen as algorithms to compute the corresponding analytic signals. In Section 3.2 a brief overview with regard to the most important non-parametric, time-frequency techniques will be provided which also includes the related signal-adaptive techniques. Thereafter, the similarities (the equivalence) of the linear transforms are mathematically demonstrated (Section 3.3). In order to achieve this, for the first time a unified time-frequency representation (TFR) is used which is based on a standardized frequency-dependent analytic signal. In Section 3.4 the WVD is introduced. The similarity conditions of the linear transforms serve as the starting point for the explanations of their differences and drawbacks, where the WVD as a quadratic transform is involved (Section 3.5). In Section 3.5.1 we compare their properties regarding the method-specific time-frequency resolution and with regard to interference (cross) term properties in Section 3.5.2. The properties will be explained by the analysis results of simulated and measured MEG data. Thereafter, the signal-adaptive approaches are described in Section 4, where the MP-based methods deal with the problems of the linear and quadratic transforms. Finally, the theoretical and application-related results are discussed (Section 5).

This review aims at a deepening of the user’s methodological knowledge in handling the above mentioned time-frequency analysis methods. To this end, the similarities, differences, their methodological subtleties, and interconnections have to be known. We attempt to demonstrate important principles step-by-step by using certain approaches while applying a mathematical description which is easily understandable. Essential fundamentals, results, and conclusions are summarized as remarks which are frequently referred to in the text. In this way, results and consequences of the mathematical investigations are described again and graphically represented by using illuminating examples, where the measured signals are derived from neuroscience research. The text and figures are complemented by videos, which illustrate the most important facts with regard to time-frequency analysis and which are available as supplementary files. It is not possible to refer to all the original articles which have contributed to the methodological development of these analyses techniques; however, we refer to the appropriate scientific publications, tutorials, and reviews which will allow the reader to deepen his/her knowledge in this field.

2. Data

The MEG/EEG data from 10 healthy volunteers (mean age 28.8 ± 5.81 years, 5 men, 5 women) were recorded in a magnetically shielded room. The biomagnetometer (Philips, Hamburg) comprises 31 parallel first order axial gradiometers with a base length of 70 mm, a diameter of 20 mm and a lateral pitch of 25 mm covering a 14 cm field of view. The overall system noise level was about 10 fT/sqrt(Hz).

The volunteers were stimulated by flicker stimuli and the frequencies were adapted to the individual’s alpha frequency. To this end the alpha frequency of each volunteer was determined during an initial resting condition of 60 s. The individualized flicker stimulations were originally conducted for 15 frequencies according to previously defined alpha-ratios ranging from 0.4 to 1.6 (αR = flicker frequency/alpha frequency). For this study we used multi-trial MEG data (20 trials) of one subject based on the stimulation with αR = 1.05. The flicker light from two white light emitting diodes was delivered via optical fibres, 90 mm away from the closed eyes of the volunteer. The intensity was far below any safety limits. Resting periods (30–60 s) were recorded between the stimulus blocks at different frequencies. The three blocks were presented in a randomized order. Within one block, the stimulation frequency was presented in a sequence of 20 trains (train = trials). A single train contained 40 flashes (pulse duration/cycle duration = 0.5; this signal has only odd harmonics) and was separated again by a resting period of 4 s [7]. The MEG was sampled at 1000 Hz and hardware filtered between 0.1 and 300 Hz and finally down-sampled to 250 Hz and shifted (in time), so that 0 s always repre-
sent the flicker onset. The recordings were carried out in the Department of Neurology (Biomagnetic Centre), Jena University Hospital. The data acquisition is shown in Figure 1 and a recording example for MEG/EEG signals can be viewed at MEG_EEG_flicker.avi.

3. Results

3.1 The Analytic Signal

We start our introduction with an apparently academic question: What is the corresponding analytic signal of an existing (measured) signal \( x(t) \)? By answering this question step-by-step, the reader will recognize a useful concept by which the nature of the linear non-parametric time-frequency techniques can be easily explained and by which the similarities and differences can be clearly differentiated.

To describe the concept of the analytic signal, the Fourier transform (FT - \( \mathcal{F} \)) is needed as a fundamental mathematical tool. By means of the FT, a signal \( x(t) \) can be represented in the frequency domain (Figure 2)

\[
X(\omega) = \mathcal{F}[x(t)](\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt
\]

where \( j \) is the imaginary unit, \( X(\omega) \) is the FT of \( x(t) \).

Remark 3.1-1

Euler’s formula \( e^{j\omega t} = \cos(\omega \cdot t) + j \cdot \sin(\omega \cdot t) \) provides a better readability of the mathematical notations and it is used throughout the text. The absolute value (\(|...|\)) of the signal’s FT \( |X(\omega)| \) is the amplitude spectrum of the signal. It provides the information about which frequency components are contained in the signal but not at which time points they occur. These fundamentals together with the solution of the integral

\[
\int_{-\infty}^{\infty} e^{j\omega t} \, dt = \frac{1}{j\omega} e^{j\omega t} + c \quad (c = 0)
\]

may help understanding of the mathematical operations and given solutions. For the explanation of linear time-frequency techniques the variable of integration \( t \) is replaced by the window time \( \tau \). The FT is also required to transform the so-called time windows \( w(t) \) into the frequency domain.

The real-valued cosine signal \( x(t) = A \cdot \cos(\phi(t)) \), with \( \phi(t) = \omega_1 \cdot t + \phi(0) \), serves as a starting point. The amplitude \( A \), the angular frequency \( \omega_1 \), and the zero phase \( \phi(0) \), i.e. the phase at \( t = 0 \), are constant parameters. The term \( \phi(t) \) is called the instantaneous phase which is (in this case) a linear function of the time with the slope of the angular frequency \( \omega_1 = \frac{2\pi}{f_1} \).

Both \( \omega \) and \( f \) are hereafter referred to as frequency and the frequency range is defined as \(-\infty < \omega < \infty \) (\(-\infty < f < \infty \)).

Here we will introduce the analytic signal which serves as a mathematically intermediate step for the description of signal properties and later for the description of the instantaneous phase which is (in this case) a linear function of the time with the slope of the angular frequency \( \omega_1 = \frac{2\pi}{f_1} \).

Both \( \omega \) and \( f \) are hereafter referred to as frequency and the frequency range is defined as \(-\infty < \omega < \infty \) (\(-\infty < f < \infty \)).

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the linear transforms. The corresponding complex analytic signal $a_x(t)$ for our real-valued cosine signal can be obtained by adding an imaginary part

$$a_x(t) = A \cdot [\cos(\varphi(t)) + j \cdot \sin(\varphi(t))] = A \cdot e^{j\varphi(t)} = A \cdot e^{j(\omega_1 t + \varphi(0))}. \quad (2)$$

One can see that $x(t) = A \cdot \cos(\varphi(t))$ is defined as its real part and the $-\pi/2$ phase-shifted cosine, i.e., $A \cdot \sin(\varphi(t))$, as its imaginary part. Both signals $x(t)$ and $a_x(t)$ have different frequency characteristics, which can be studied after a transform into the frequency domain. Before the FT is carried out, we will introduce a multiplication of both signals with a rectangular function $1(t)$ (Figure 2 upper row). This procedure is called windowing. The rectangular window with the duration $T$ can be described by

$$1(t) = \begin{cases} 1 & \text{for } -T/2 \leq t \leq T/2, \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

where the FT of the window yields

$$X_1(\omega) = \int_{-T/2}^{T/2} 1(t) \cdot e^{-j\omega t} \, dt = T \cdot \frac{\sin(\omega T/2)}{\omega T/2} = T \cdot \text{sinc} \left( \frac{\omega T}{2} \right). \quad (4)$$

The sinc-function is defined by $\text{sinc}(y) = \sin(y)/y$. For the calculation of the FT of the windowed signal $a_x(t) \cdot 1(t)$ and of its analytic signal $a_x(t) \cdot 1(t)$, the bounds of integration can be limited to $-T/2$ and $T/2$. The results of the FT of the real cosine signal and of its analytic signal (Eq. 2) are depicted in Figure 2. Already by this simple example five remarks 3.1-2–3.1-6, which are important for the understanding of the time-frequency techniques, can be derived.

**Remark 3.1-2**

The real-valued cosine signal is represented in the frequency domain by mirrored peaks at $\pm \omega_1$. The FT of the analytic signal only yields a peak at the positive frequency $\omega_1$ with doubled amplitude. This means that the analytic signal can be defined as a signal with spectral components for positive frequencies only. Gauss windowed analytic signals are shown in Figure 4, which are called Gabor or Morlet wavelets and their FT leads to results for positive frequencies only.

**Remark 3.1-3**

Additionally, it can be generalized that the time windowing of a signal results in a peak’s shape which is determined by the window’s FT [8]. For infinite bounds of integration (infinite rectangular window), only lines at $\pm \omega_1$ with the amplitude $A$ would occur. The rectangular window produces sinc-shaped peaks. By using a Gaussian time window, Gaussian peaks are obtained, because the FT of a Gaussian window

$$w_G(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/(2\sigma^2)}$$

yields a Gaussian function in the frequency domain (Gaussian frequency window)

$$X_G(\omega) = \mathcal{F} \left[ \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/(2\sigma^2)} \right](\omega) = e^{-\frac{\omega^2\sigma^2}{2}}$$

which provides Gaussian peaks at $\pm \omega_1$ for

$$X(\omega) = \frac{A}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\omega-\omega_1)^2}{2\sigma^2}} + \frac{A}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\omega+\omega_1)^2}{2\sigma^2}}$$

and at $\omega_1$ for

$$a_{X}(\omega) = A \cdot e^{-\frac{(\omega-\omega_1)^2}{2\sigma^2}} \cdot \text{sinc} \left( \frac{\omega T}{2} \right). \quad (6)$$

Windowing leads to infinite peaks, i.e. the frequency range in Equations 4 and 5 is
defined by \(- \infty < \omega < \infty\). However, in practical signal analysis the \(w_c(t)\) must be truncated at the ends by using a window’s time-range of \(2\) (or \(3\)) \(\cdot \sigma\). A variety of other functions for windowing is known, however, seldom applied in biomedical signal analysis.

**Remark 3.1-4**

According to \(\text{Figure } 4\), both Gaussians are characterized by a reciprocal relationship of the quantities

\[
\sigma_\omega = 1/\sigma_t, \quad \sigma_t \cdot \sigma_\omega = 1
\]  

(7)

The quantity \(\sigma\) determines both inflection points \((\pm \sigma)\) of a zero-mean Gauss function on the abscissa (known as standard deviation in statistics) and is appropriate to describe the width of the function (width parameter). From the reciprocal relationship of both width parameters it can be intuitively derived that a broad Gaussian time-window will produce a narrow Gaussian peak in the spectrum and vice versa. Using a broad time window, frequency components can be detected and separated very well due to the resulting narrow peaks, i.e. a high frequency resolution can be stated. In contrast to this, at which time points these frequency components occur is unprecise (it occurs within the broad window), i.e. there is a low time resolution. The Gaussian window minimizes the so-called time-bandwidth product (TBP), i.e. it optimizes the conflicting constraints of time- and frequency-resolution. For Gaussians the TBP can be given by \(\sigma_t \cdot \sigma_\omega = 1\) for other windows the Heisenberg uncertainty TBP \(\geq 1\) holds.

**Remark 3.1-5**

\(\text{Figure } 2\) depicts a way in which the analytic signal can be computed. The FT of a real (measured) signal has to be performed. Thereafter the negative frequencies are set to zero and complex-valued FT results at positive frequencies will be doubled. The inverse FT \((\mathcal{F}^{-1})\) of the modified result of the signal’s FT generates the corresponding analytic signal. This procedure can be mathematically expressed by

\[
\tilde{x}(t) = \mathcal{F}^{-1} [2 \cdot \mathcal{F}[x](\omega) \cdot \varepsilon(\omega)](t)
\]  

(8)

where \(\varepsilon(\omega)\) is the unit step function

\[
\varepsilon(\omega) = \begin{cases} 
1 & \text{for } \omega > 0 \\
0 & \text{for } \omega \leq 0
\end{cases}
\]  

(9)

An additional frequency band limitation (filtering) can be introduced for positive frequencies, i.e. a band pass limitation by setting all frequencies to zero except for the pass band frequencies \(\omega_l \leq \omega \leq \omega_u\) (between the lower and the upper cut-off frequency), which is identical to a multiplication with a rectangular window in the frequency range

\[
1(\omega) = \begin{cases} 
1 & \text{for } \omega_l \leq \omega \leq \omega_u \\
0 & \text{otherwise}
\end{cases}
\]  

(10)

If the rectangular frequency window is replaced by a Gaussian frequency window function \((\text{Figure } 11)\) then the inverse FT provides a band pass filtered analytic signal which is windowed by the corresponding Gaussian time window. The location of the maximum of the Gaussian frequency window is the center frequency \(\omega_c\) and its \(\sigma_c\) is a measure for the bandwidth of the filter function. Correspondingly, \(\text{Figure } 8\) can be written as

\[
\tilde{x}(t, \omega_c) = \mathcal{F}^{-1} [2 \cdot \mathcal{F}[x](\omega) \cdot G(\omega_c, \omega)](t)
\]  

(11)

with \(G(\omega_c, \omega) = \varepsilon_{\text{rect}}(\omega)\). The Gaussians in \(\text{Figure } 4\) can also be seen as \(t\) of Gaussian time windows \(\text{(A)}\) if the center frequencies in \(\text{(C)}\) are set to zero.

**Remark 3.1-6**

Windowing leads to infinite spectral peaks (remark 3.1-3), which in turn means that an overlapping to some extent of both mir-
rered peaks at ± ω1 occurs, i.e. the positive frequency range, which is used for the calculation of the analytic signal (\(\text{Eq. 8}\)), is interfered by one tail of the peak at −ω1. Consequently, only an approximated analytic signal can be calculated for windowed signals. One can only decrease interference by using a high frequency resolution, i.e. broad time windows. Theoretically, an infinite rectangular time window would avoid an overlap because peaks will collapse to lines at ±ω1. For a two-component signal \(x(t) = A_1 \cdot \cos(\omega_1 t) + A_2 \cdot \cos(\omega_2 f)\) (\(\phi_{1/2}(0) = 0\)) two mirrored peak configurations at ±ω1 and ±ω2 occur. Now, in the positive frequency range both peaks at ω1 and ω2 will more or less overlap, in particular in the intermediate frequency range. This interference is stronger than the interference which is originated from the more distant peaks at negative frequencies. The lower the frequency distances |ω2 − ω1| the more contaminations will occur, i.e. the calculation of the analytic signal will be distorted. This is also true for the selection of one frequency component by band pass filtering according to \(\text{Eq. 11}\).

The remarks 3.1.2–3.1.6 are related to sinusoidal signals with constant parameters. Now, we will proceed towards time-varying signal characteristics by simple modifications of our cosine signal. Firstly, the constant amplitude A is replaced by a time-dependent amplitude \(A(t)\), i.e. an amplitude modulation \(x(t) = A(t) \cdot \cos(\phi(t))\) with \(A(t) = c_1 + m_{1}(t)\) is introduced. According to the rules of amplitude modulation, the amplitude of \(m_{1}(t)\) must be lower than the value of the constant \(c_1\) and its highest frequency must be ≤ ω/2. Via the calculation of the corresponding analytic signal (\(\text{Eq. 8}\)) one can extract \(A(t)\) from \(x(t)\) as the instantaneous amplitude \(\hat{A}(t)\) (envelope) according to

\[
\hat{A}(t) = |\hat{x}(t)| = \sqrt{\left(\text{Re}(\hat{x}(t))\right)^2 + \left(\text{Im}(\hat{x}(t))\right)^2} \tag{12}
\]

where \(\text{Re}\) designates the real part and \(\text{Im}\) the imaginary part. As mentioned above, the signal \(x(t)\) can be used as the real part. This procedure describes a demodulation of the amplitude-modulated carrier signal \(c_1 \cdot \cos(\omega_1 t + \phi(t))\) and the envelope can be seen as the time-variant amplitude spectrum of the carrier signal.

**Remark 3.1.7**

A popular model is the amplitude modulation by a cosine \(A(t) = c_1 + c_A \cdot \cos(\omega_A t)\) with \(\omega_A << \omega_1\) and \(c_A \leq c_1\), which provides a spectrum with a triple peak configuration at \(\omega_1\) (carrier band) and at \(\omega_1 - \omega_A, \omega_1 + \omega_A\) (side bands).

Secondly, the frequency of the cosine signal should be modulated. The instantaneous phase \(\phi(t)\) becomes different from the linear slope \(\omega_1\) when an additional time function \(c_2 \cdot m_2(t)\) as modulation term \((c_2\) is a constant) is included

\[
\phi(t) = (\omega_1 + c_2 \cdot m_2(t))t + \phi(0). \tag{13}
\]

And the cosine’s “frequency” instantaneously changes according to the modulation function. In such a case the notion of an instantaneous frequency \(\dot{\phi}(t)\) makes sense which can be determined by means of the analytic signal

\[
\dot{\phi}(t) = \frac{1}{2\pi} \frac{d}{dt} \text{Im}(\hat{x}(t)) \tag{14}
\]

with \(\text{Im}(\hat{x}(t)) = \arg(\hat{x}(t)). \tag{15}\)

\(\dot{\phi}(t)\) is the calculated instantaneous phase. For simplicity we assume that the frequency modulation of the carrier signal is small and that the highest frequency of \(m_2(t)\) is much lower than \(\omega_1\). Instead of the frequency modulation a phase modulation can be carried out by adding a time-variant function \(\phi(t) = \omega_1 t + m_{1b}(t) + \phi(0)\). However, each frequency modulation is accompanied by a phase modulation and vice versa as shown by \(\text{Eq. 14}\). \(\dot{\phi}(t)\), \(\dot{\phi}(t)\) can be seen as the resulting (calculated) signals after a frequency/phase demodulation of the signal.

By inserting \(m_1(t)\) and \(m_2(t)\) into the cosine signal an amplitude and frequency/phase modulated signal can be obtained, where both modulations have the same carrier frequency \(\omega_1\). By such a signal a narrow-banded signal (index NB) can be approximated. The corresponding analytic signal

\[
\hat{x}_{NB}(t) = A(t) \cdot e^{i\phi(t)} \tag{16}
\]

can only properly provided when the spectrum of \(A(t)\) (baseband of the amplitude modulation) can be clearly distinguished from the spectrum of \(\phi(t)\) \([5]\). This is true as long as the amplitude and the frequency are slowly varying. Consequently, the more a signal is narrow-banded the more likely the calculation of the analytic signal provides an accurate model of a real system and the better in general the calculation of the \(\dot{\phi}(t)\) will be \([5]\).

For this review we define such a narrow-banded signal as a *mono-component signal*. However, an analytic signal can also be calculated for a sum of a number of narrow-banded (mono-component) signals (according to \([5]\), p. 58)

\[
x(t) = \sum_{i=1}^{\infty} A_i(t) \cos(\phi_i(t)) + A_2(t) \cos(\phi_2(t)) + ... \tag{17}
\]

where the analytic signal can be given by

\[
\hat{x}(t) = \sum_{i=1}^{\infty} A_1(t) e^{i\phi_1(t)} + A_2(t) e^{i\phi_2(t)} + ... \tag{18}
\]

As described for two time-invariant components (remark 3.1–6), interference of the component’s spectra will occur, i.e. a selection of one component by a narrow-band filter (\(\text{Eq. 11}\)) will also pick up interference terms from other components. Therefore, it would be desirable that the components in \(\text{Eq. 17}\) are narrow-banded and/or the distances between the carrier frequencies (or center frequencies of the band) are large. In such cases the calculations according to \(\text{Eq. 8}\) approximately produce an analytic signal for these specific multi-component signals. However, an amplitude and frequency/phase demodulation of such a signal would lead to instantaneous amplitude and frequency courses without physical meaning. Nevertheless, analytical signals of the signal class given by \(\text{Eq. 17}\) are used to enhance the WVD approach.

Until now the analytic signals which have been introduced have been frequency-independent, with the exception of the
band pass filtered analytic signal (Eq. 11), i.e. their application for the calculation of time-variant signal properties is restricted to mono-component signals. However, for a time-variant signal analysis which is suited for broad application, a frequency-dependent analytic signal definition is required. Consequently, methods are appropriate which decompose a signal into frequency-dependent analytic signals \( \psi_x(t, \omega) \). This can be performed by using the linear transforms STFT/GT, ST, and CMWT (Figure 3), i.e. these methods provide the analytic signals for each frequency. The frequency-dependent analytic signal is in the joint space of time and frequency and is almost identical to a time-dependent (or time-variant) spectrum of the signal.

In fact, time and frequency are the two independent variables of a complex valued analytic signal, also known as time-frequency representation (TFR).

The instantaneous amplitude (envelope) and phase can be given as frequency-dependent measures analogously to Equations 12, 14 and 15

\[
A(t, \omega) = |\psi_x(t, \omega)| \quad (19)
\]

\[
\psi(t, \omega) = \arg \psi_x(t, \omega) \quad (20)
\]

\[
\phi f(t, \omega) = \frac{d}{dt} \psi(t, \omega) \quad (21)
\]

where \( A(t, \omega) \) represents the time-variant amplitude spectrum and

\[
P(t, \omega) = |\psi_x(t, \omega)|^2 \quad (22)
\]

the time-variant power spectrum or the quadratic representation of the time-frequency characteristics of the signal.

The frequency-dependent instantaneous phase \( \psi(t, \omega) \) can be utilized for the definition of several time-variant phase-locking and phase-synchronization measures [9]. The time-variant power spectrum \( P(t, \omega) \) representation derived from STFT/GT and ST is also known as a spectrogram and is derived from CMWT as a scalogram. By means of these approaches time-variant multi-component signals can be analyzed. As we will see in Section 3.3, the HT represents a special case because this procedure provides a frequency-independent analytic signal, i.e. the frequency-dependency can only be achieved by using a band pass filtering.

### 3.2 Classification of Time-frequency Techniques

The most frequently used time-frequency techniques in biomedicine, apart from the time-variant parametric approaches, are the spectrogram (GT, ST), scalogram (CMWT), and TFD (WVD).

The linear transforms are the short-time Fourier transform (STFT) and Gabor transform (STFT by using a Gaussian window), S-transform (ST), continuous Morlet wavelet transform (CMWT) and Hilbert transform (HT).

In the following section we will show that these transforms are equivalent when specific windows/kernels/filters are used. The HT can be applied for the computation of the instantaneous amplitude (envelope) and frequency of a mono-component signal via its corresponding analytic signal. As mentioned above, the concept of the analytic signal can also be used to explain the computation of the time-frequency representations of the other linear transforms which are suitable for the analysis of multi-component signals.

Additionally, the analytic signal can be beneficially used, instead of the original signal, for the computation of quadratic TFRs [5]. The spectrogram (square magnitude of STFT/GT and ST) and the scalogram (square magnitude of the CMWT) can also be assigned to TFDs, the spectrogram to the Cohen’s class (designation \( \mathcal{D} \)) and the scalogram (designation \( \mathcal{G} \)) to the affine-invariant TFDs. The Wigner-Ville distribution (WVD) is the most prominent element of the Cohen’s class and by means of the smoothed-pseudo WVD (SPWVD) a continuous passage from the spectrogram to the WVD can be achieved by changing smoothing and/or window functions. The affine SPWVD allows a continuous passage from the scalogram to the affine WVD. These interconnections and their conditions are described by Auger et al. [10]. The affine TFDs are not further discussed in this article.

The linkages and passages between these techniques show that the advantages and disadvantages of any method must be weighed before application, which requires a priori knowledge with regard to the signal characteristics.

The GT, HT and WVD are beneficially used in combination with decomposition algorithms. Such combinations allow a more adaptive time-frequency analysis. The matching pursuit (MP) approach (combination of the MP decomposition with the WVD) was introduced by Mallat and Zhang [11] and the matched Gabor transform (MGT) (combination of MP with GT) by Wacker and Witte [12]. The latter approach was developed for adaptive phase extraction from the signal. The MP is a so-called atomic decomposition method (e.g. Gabor atom dictionary) which iteratively decomposes the signal into an approximation of a sum of complex Gabor waves (see Section 4). The Gabor wavelet (Figure 4) is the central element of the GT, ST, CMWT, and the MP (with Gaussian dictionaries). In the following we will use the term “wavelet” in connection with a transform and the term “atom” in connection with the MP decomposition. Gabor wavelets/atoms can be obtained by a multiplication of a Gaussian window \( w_G \) and the analytic signal with a frequency \( \omega_c \) (with \( A = 1 \))

\[
g(t, \omega_c \sigma, \tau) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\tau^2}{2\sigma^2}} \cdot e^{i\omega_c \sigma t} = w_G \cdot (\cos(\omega_c t) + j \cdot \sin(\omega_c t) \cdot e^{-\frac{\tau^2}{2\sigma^2}} \cdot e^{-j\omega_c \tau}) \quad (23)
\]

The mirrored Gabor wavelet/atom by \( (t \text{ replaced by } -t) \) can be given

\[
g(t, \omega_c, \sigma) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\tau^2}{2\sigma^2}} \cdot e^{-j\omega_c t},
\]

is the complex conjugate of \( g(t, \omega_c, \sigma) \).
In 1998 Huang et al. [13] introduced the empirical mode decomposition (EMD) in combination with the HT which was later known as Hilbert-Huang transform (HHT). The EMD iteratively decomposes a multi-component signal into a set of mono-component signals which can be analyzed by means of the HT.

The following section will focus on the similarities and differences between the linear transforms. Using the concept of the analytic signal will lead to a unified framework for complex time-frequency information so that a distinction between a frequency-dependent analytic signal and a time-variant spectrum (spectrogram/scalogram) will be clear afterwards.

### 3.3 similarities between the Linear Transforms

The FT gives frequency information but the spectrum is independent of the time. So if one wants to use the FT for time-frequency analysis, the straightforward concept is to use a sliding window to localize the signal information and thus also localize the frequency information. The STFT can be written as

\[ \text{STFT}(t, \omega) = \int_{-\infty}^{\infty} x(\tau) \cdot w(\tau - t) \cdot e^{-j\omega \tau} d\tau \]

\[ \mathcal{F}[x(\tau) \cdot w(\tau - t)](\omega). \]

where \( w(\tau) \) is a window function which pre-windowed the signal around a particular time (Figure 2) and which slides over the absolute time \( t \). For each instant of the absolute time \( t \), the FT is carried out over window time \( \tau \). It is assumed that the windowed signal is time-invariant (stationary). The result is a function of the time \( t \), which is the shifting time for the window, and of the frequency \( \omega \). The FT of the windowed cosine signal in Figure 2 (left column) can be seen as the realization of the STFT for one instant of time during shifting using a rectangular window \( 1(t) \).

The time-frequency properties of the STFT heavily depend on the chosen window. Thus depending on the FT of the window (rectangular, Bartlett, Hann, Hamming, Blackman, Gauss etc.) the shape of the time-variant spectrum can change significantly. In addition to the rectangular window, the Gaussian window is the most commonly used window function. The STFT which uses a sliding Gaussian time window

\[ w_{GT}(t - \tau) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(\tau - \omega)^2}{2\sigma^2}} \]

(25)

is called GT. However, several authors use the term GT as a synonym for STFT, i.e. independent from the window used.

Before introducing the analytic signal of the GT \( \phi_{GT}(\tau, \omega) \), we multiply \( \text{STFT}(t, \omega) \) with a frequency-dependent phase correction term \( e^{j\omega \tau} \) (no contribution to the amplitude because \( |e^{j\omega \tau}| = 1 \)) to achieve its comparability with those derived from the other transforms, i.e. we used a comparable, standardized analytic signal for all transforms.

\[ a(x(t, \omega)) = e^{j\omega t} \cdot \text{STFT}(t, \omega). \]

(26)

Trivially, we use the same term for the GT as it is a special case of the STFT

\[ a(x(t, \omega)) = e^{j\omega t} \cdot \phi_{GT}(t, \omega). \]

(27)

where the GT and with it the analytic signal can be given by

\[ \phi_{GT}(t, \omega) = \int_{-\infty}^{\infty} x(\tau) \cdot \frac{1}{\sqrt{2\pi \sigma^2}} e^{j\omega \tau} \cdot e^{-j\omega t} d\tau \]

(28)

The standardized analytic signal according to Equation 27 can be rewritten

\[ a(x(t, \omega)) = \int_{-\infty}^{\infty} x(\tau) \cdot \frac{1}{\sqrt{2\pi \sigma^2}} e^{j\omega \tau} \cdot e^{-j\omega t} \cdot \frac{e^{-j\omega t} \cdot \ast g}{\| \ast g \|^2} d\tau \]

(29)

The integral (29) is the convolution integral, which describes the GT as a filtering of the signal \( x(t) \) in the time domain by a filter bank in which the complex conjugate Gabor wavelets \( g_{GT}(t, \omega, \sigma) \) (one for each frequency) are used as complex filter weights. The term \( w \) designates the window function used. Using the convolution notation (“∗” convolution operator) the standardized analytic signal can be written as

\[ a(x(t, \omega)) = x(t) \ast g_{GT}(t, \omega, \sigma), \]

(30)

Remark 3.3.1

The mirroring of the filter weights is a part of the convolution (filtering) operation, where the mirroring of a Gabor wavelet \( g_{GT}(t, \omega, \sigma) \) (Equation 23) yields the complex conjugate one \( \overline{g_{GT}(t, \omega, \sigma)} \). This is also valid for the ST and CMWT which also uses Gaussian windowed wavelets. The filtering of the signal in the time domain by \( g_{GT} \) (Equation 30) corresponds to the filtering in the frequency domain by means of a frequency window \( \mathcal{F}[g(t, \omega, \sigma)](\omega) \) (Equation 11). This relationship points to different implementation modalities. In addition to the STFT’s sliding window modality, implementations as a filter bank in time (Equation 30) or frequency domain (Equation 11) are most frequently used for GT, ST, and CMWT.

The GT uses a constant width parameter \( \sigma \) for the Gaussian window to keep a constant time window for the analysis (Figures 4a and 4b), i.e. for a low frequency signal component the Gabor wavelet is composed of a fewer number of oscillations (Figure 4Ba) than for a high frequency component (Figure 4Bb).

Dependent on the analysis goal and data properties, it might be beneficial to equalize the number of oscillations in the time window which implies the frequency-dependent variation of the width parameter \( \sigma \) for the Gaussian window (Figures 4Ba and 4Bc). For this aim the ST was introduced by Stockwell et al. [14] in 1996 and the corresponding analytic signal can be computed using a frequency-dependent Gaussian window \( w_{GT}(t - \tau) \) for the FT formalism

\[ \phi_{ST}(t, \omega) = \int_{-\infty}^{\infty} x(\tau) \cdot \frac{1}{\sqrt{2\pi \sigma^2}} e^{j\omega \tau} \cdot e^{-j\omega t} d\tau \]

(31)

The width parameter of the Gauss function (Equation 23) is reciprocal to the frequen-
\[ a(t) = e^{i \omega t} \cdot \frac{\sin(\omega t)}{\omega t} \]

The analytic signals for a certain frequency \( \omega \), computed by the GT and by the ST, are identical if the parameter in the GT is chosen to be \( \sigma_i = \frac{2\pi}{\omega} = \frac{1}{f} \).

The above mentioned idea to have an equal number of oscillations in the window for every frequency comes from the concept of multi-scale analysis (in contrast to multi-frequency analysis). In the past decades, the multi-scale analysis has been inevitably linked to wavelets. Wavelet analysis involves use of a base waveform called a mother wavelet with the frequency \( \omega_0 \) for the entire analysis; all other wavelets are obtained by simply scaling the mother wavelet. The Morlet wavelet is not only one of the most popular complex wavelets, it is also the one for which the formula and results are very close to those of the GT, or even closer to the ST. Thus the CMWT is the only wavelet transform that is considered in this overview. In our definition of the Morlet mother wavelet it is only parameterized by \( \omega_0 \) and can be approximated by

\[
\Psi(t) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} e^{i \omega_0 t},
\]

from which a generalized frequency-dependent wavelet can be derived by introducing a scaling parameter \( s \) and replacing \( t \) by \( t/s \) which results in a scaled frequency \( \omega = \omega_0/s \)

\[
\Psi(t, \omega) = \frac{1}{\sqrt{\pi}} e^{-\frac{t^2}{2\sigma^2}} e^{i \omega t}.
\]

Other authors (e.g. [15], p. 139) use a more general notation of the Morlet mother wavelet in which a control parameter for an independent choice of its time-resolution is usually added. In order to obtain invariance regarding the WT and its inverse counterpart, normalization factors have to be used, which can be distributed in different ways between the forward and inverse transform. Note however that a symmetric distribution of \( 1/\sigma \) for both directions is most commonly used, because the resulting convolution kernels have a scale-constant \( L_1 \)-norm. With that, the CMWT can be written as follows

\[
w_{\omega} x(t, \omega) = \int_{-\infty}^{\infty} x(t) \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-t)^2}{2\sigma^2}} e^{i \omega (t-t)} dt.
\]

In order to achieve normalized amplitude information (scale-constant \( L_1 \)-norm instead of \( L_1 \)-norm) a multiplication with \( 1/\sigma \) is performed, which means that the full normalization factor of forward and inverse transform are applied in the forward transform. Carrying out the computation yields

\[
w_{\omega} x(t, \omega) = \int_{-\infty}^{\infty} x(t) \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-t)^2}{2\sigma^2}} e^{i \omega (t-t)} dt = \sqrt{\frac{\sigma}{2\pi}} \int_{-\infty}^{\infty} x(t) \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-t)^2}{2\sigma^2}} e^{i \omega (t-t)} dt.
\]

The standardized analytic signal of the CMWT can be obtained by

\[
a_{\omega} x(t, \omega) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\Psi(t, \omega)}{\Psi^*}(t) dt = \int_{-\infty}^{\infty} x(t) \frac{1}{\sqrt{2\pi}} e^{-\frac{(t-t)^2}{2\sigma^2}} e^{i \omega (t-t)} dt.
\]

The CMWT has a frequency-dependent time-frequency resolution according to Figures 4a and 4c, which illustrates the dependency for two defined frequencies.

\[
\mathcal{F}[b(t)](\omega) = \frac{1}{e^{\frac{(\omega-\omega_0)^2}{2\sigma^2}}} - e^{\frac{(\omega-\omega_0)^2}{2\sigma^2}}.
\]

Remark 3.3-2

GT, ST, and CMWT use Gaussian windows \( w_{GT,ST,CMWT}(t) \), i.e. Gabor/Morlet wavelets \( w_{GT,ST,CMWT}(t, \omega, \sigma) \). The principles of the GT and CMWT analysis can be viewed at Gabor_transform.avi and Morlet_transform.avi which are available as supplementary files, connection with the results in Figure 4B, only the black cosine part. The double-sided transfer function of such a filter is

\[
\mathcal{F}[b(t)](\omega) \approx \frac{1}{2} \left[ \frac{1}{e^{\frac{(\omega-\omega_0)^2}{2\sigma^2}}} - e^{\frac{(\omega-\omega_0)^2}{2\sigma^2}} \right] ^2.
\]

If the transfer function is multiplied with \( e(\omega) \) given in Equation 9, the influence of the Gaussian function at \( -\omega_0 \) is suppressed to a large extent so that one can approximate the result quite well by a single Gauss function at \( \omega_0 \)

\[
\mathcal{F}[b(t)](\omega) \cdot e(\omega) \approx \frac{1}{2} e^{\frac{-(\omega-\omega_0)^2}{2\sigma^2}}.
\]
mechanics by Eugene P. Wigner and discussed in the framework of signal analysis by J. Ville in 1948, shortly after (in 1946) Dennis Gabor defined time-frequency methods. Claassen and Mecklenbräuker [18] provided a way to define a time-dependent energy spectrum for the discrete-time version of the WVD and compared it with other approaches [19]. Although the WVD has good theoretical properties, the major drawback is that it can produce interference terms (also called cross terms) between components of a multi-component signal, i.e. additional components in the time-frequency plane are indicated which do not exist (Figure 5). This fact will be discussed later. To partially reduce the interference terms, it is usual to use the analytic signal instead of the signal itself, i.e.

\[
W(x(t), \omega) = \int_{-\infty}^{\infty} s(t - \tau/2) \cdot A(t + \tau/2) \cdot e^{-j\omega \tau} d\tau,
\]

where \(s(t + \tau/2)\) is the backward shifted analytic signal of \(s(t)\) and \(A(t - \tau/2)\) is the corresponding forward shifted complex conjugate analytic signal. The analytic signal can be calculated according to Equation 8 which is equivalent to the HT approach. For the WVD the FT of a temporal covariance (CV) function for all time points \(t\) is carried out, i.e. the FT is acting on the delay time \(\tau\) for which the covariance at \(t\) is calculated. CV(\(t, \tau\)) is symmetrized with respect to the evaluation time \(t\). The definition of the WVD (Equation 45) requires knowledge of the CV(\(t, \tau\)) from \(\tau = -\infty\) to \(\tau = \infty\) which can be a problem in practice [10]. Therefore, a time-windowed version, the pseudo-WVD (PWVD) is introduced which decreases the frequency resolution (time-windowing is equivalent to a frequency smoothing, see Figure 2) and the manifestation of interferences. By adding a separable smoothing window an independent control of smoothing in time and frequency are offered, i.e. the time- and the frequency resolution can be controlled independently. This version is most frequently used in signal analysis and known as smoothed PWVD (SPWVD). The key properties of the WVD will be discussed in comparison with those of the linear transforms in the following section. We would like to highlight here that the WVD does not provide instantaneous phase information, i.e. the WVD cannot be utilized for e.g. time-variant synchronization analysis.

3.5 Differences between Time-frequency Techniques

3.5.1 Time-frequency Resolution

Time-frequency resolution is probably the least understood element in time-frequency analysis and yet it is the most important. When interpreting results, it is mandatory to understand the effects of time-frequency resolution. The basic message given by linear transforms was that a broadening of the time-window (decrease of the time-resolution) increases the frequency resolution and vice versa. The fundamental equation for time-frequency resolution is the Heisenberg uncertainty principle. It states that localization in time and localization in frequency are contradictory aims and that the TBP has a lower bound, which is achieved by Gabor wavelets (the wavelet type for GT, ST, and CMWT). The quantity \(\sigma_o\) and with it also \(\sigma_w\), can be used to characterize their time-frequency resolution (Figure 4).

Table 1 shows an overview of the time-frequency resolutions of the linear transforms according to the formal derivation of the unified representation in Section 3.3. All effects that will be demonstrated in the following can easily be explained with these formulas. A visualization is attempted of what can happen in time-frequency analysis. We will only show results for the GT, CMWT and WVD. The general properties of the STFT depend on the chosen window, the properties of the HT depend on the chosen band pass filter and the properties of the ST are included in the analysis of the CMWT for \(\omega_m = 2\pi\).

Figure 5 demonstrates the effects of different time-frequency resolutions by applying GT and CMWT to three exemplary test signals. Two different parameter settings (referred as GT1/2 and CMWT1/2) were used. Additionally, the WVD using the analytic signal is applied to the test signals. The three signals are presented in the first row of the image lattice. The spectro-
grams/scalograms/TFDs are represented as maps in which the power is expressed by different shades of gray.

In column A, a Dirac impulse (unit impulse) is analyzed in order to illustrate the time-resolution characteristics of the transforms. Theoretically, the Dirac impulse (Figure 5Aa) is characterized by a time-invariant constant spectrum over the whole frequency range, i.e. the time-variant spectrum should reveal a constant amplitude for each frequency at the time of occurrence of the impulse, i.e. a vertical line in the time-frequency plane (Figure 5Ab to Af) should result. This is only true by analyzing the impulse by the WVD. However, if the analytic signal of the Dirac impulse is analyzed by the WVD (as used here), then this optimal property gets lost (Figure 5Af). However, the use of the analytic signal partially reduces interference terms in the TFD of other signals (Eqs. 16 and 17). For GT the frequency-independent time resolution results, where the power is distributed over time according to the chosen Gaussian window, i.e. according to \( \sigma_t \) (GT1: \( \sigma_t = 0.1 \) s, GT2: \( \sigma_t = 0.05 \) s). The frequency-dependent time resolution of the CMWT results in a decreasing power-blurring over time with increasing frequency and vice versa. This was shown for two Morlet mother wavelets with the frequencies \( \omega_01 = 4\pi \) (Ad) and \( \omega_02 = 2\pi \) (Ae). According to Equation 33, at both frequencies an identical time resolution is obtained.

Column B provides a superposition of two stationary oscillations according to the example in remark 3.1-6 with \( f_1 = 2 \) Hz and \( f_2 = 15 \) Hz. This test signal serves to illustrate the transform’s frequency resolution characteristics because the signal properties are time-invariant (stationary), i.e. actually no time resolution is needed. With GT1 (low time resolution) the stationarity of the signal is reproduced in the spectrogram (Figure 5Bb) with a sufficient frequency resolution. With higher time resolution (GT2), i.e. low frequency resolution, the separation of both oscillations is possible but weak interferences occur which oscillate with a frequency of \( f_2 - f_1 = 13 \) Hz. For CMWT1 (low time and high frequency resolution) the stationary oscillations are adequately represented in the Morlet scalogram (Figure 5Bd), where the high-frequency component is characterized by a sufficient frequency resolution and the low-frequency component by an optimal frequency resolution (horizontal line, no frequency blurring). CMWT2 uses a mother wavelet with higher time resolution for both oscillations which results in a lower frequency resolution for both (Figure 5Be). The WVD’s TFD shows a high time and frequency resolution (Figure 5Af and 5Bf) and an optimal separation of both oscillations. The stationary character of the oscillations is also reproduced. The disadvantage is obvious, the occurrence of strong oscillating interference term between both frequency components at 6.5 Hz which oscillates with 13 Hz, i.e. a component is represented which does not exist. The properties of the interference terms will be discussed in Section 3.5.2.

Finally, column C of Figure 5 shows the results for an oscillation at 15 Hz whose

**Table 1** Outline with regard to the time-frequency resolution characteristics of linear transforms

<table>
<thead>
<tr>
<th>Transform</th>
<th>( \sigma_t, \sigma_w )</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>STFT</td>
<td>( \sigma_t &gt; 1/\sigma_w )</td>
<td>the quantities depend on the window, ( \sigma_t (\sigma_w) ) is freely selectable</td>
</tr>
<tr>
<td>GT</td>
<td>( \sigma_t = 1/\sigma_w )</td>
<td>( \sigma_t (\sigma_w) ) is freely selectable</td>
</tr>
<tr>
<td>ST</td>
<td>( \sigma_t = 2\pi/\omega = 1/\sigma_w )</td>
<td>( \sigma_t (\sigma_w) ) is bound to the frequency</td>
</tr>
<tr>
<td>CMWT</td>
<td>( \sigma_t = \omega_0/\omega = 1/\sigma_w )</td>
<td>( \omega_0 ) is freely selectable, ( \sigma_t (\sigma_w) ) is bound to the frequency and ( \omega_0 ) (see Section 5, choice of the Morlet mother wavelet)</td>
</tr>
<tr>
<td>HT</td>
<td>( \sigma_t (\sigma_w) )</td>
<td>quantities depend on the band pass filtering, can be similar to GT</td>
</tr>
</tbody>
</table>

Figure 5 Comparison of the results of spectrogram/scalogram/TFD analyses by using GT (GT1 and GT2, b and c) and CMWT (d and e) by with different time-frequency resolutions as well as the WVD (f). Three test signals are used (a), the Dirac impulse (Aa), a superposition of two stationary oscillations (Ba), and two subsequent Gaussian-shaped transient oscillations (Ca).
amplitude is modulated by two Gaussians, where the first is centered at 0.25 s and the second is centered at 0.75 s, two subsequent transient (non-stationary) spindle-like oscillations occur. By using GT1 both spindles cannot be exactly separated in the spectrogram (Figure 5Cb) because the low time resolution favors the stationary nature of the signal, i.e. the amplitude modulation with a carrier peak and two side bands (see remark 3.1-7). GT2 has a more optimal time resolution for the analysis of the spindles, indicated by the exact location of the spindles’ maxima in time and the clear separation of both (Figure 5Cc). CMWT1 has a similar time-frequency resolution for the transient 15-Hz-spindles as GT1. Consequently, the scalogram (Figure 5Cd) indicates an amplitude modulation with an almost stationary three-peak configuration over the whole analysis time. The frequency resolution is sufficient enough for a peak separation. With higher time resolution (CMWT2) the transient nature of the spindle signal is more accentuated (Figure 5Ce). Due to the frequency-dependent time-frequency resolution the spindle related peaks are unsymmetrically blurred. The WVD provides a representation of the spindles with a good time and frequency resolution (Figure 5Cf). Between both spindles interference terms occur which strongly hamper the interpretation of the TFD. In Auger et al. [10] similar spindle signals are used to illustrate the interference locations in time and frequency.

Remark 3.5.1-1
From these exemplary time-frequency analyses the main advantages and disadvantages of the approaches can be derived. Clearly the choice of method and its time-frequency resolution (GT, CMWT) must be based on a priori knowledge of the time-frequency signal characteristics as well as consideration of the aims of the analysis; thus which method is chosen is driven by the working hypothesis. Strictly speaking, the relative performance of the various time-frequency approaches depends on the signal [20]. One can simply conclude that GT, ST, and CMWT work best if the analysis wavelets used match the transient signal components that are to be analyzed, i.e. that an optimal manually chosen match can only be obtained for one wavelet defined by its envelope and frequency.

In Figure 6 the results of MEG spectrogram/scalogram/TFD and phase-locking analyses for one subject are shown. The estimations are based on 20 trials of repetitive flicker stimulation which start at 0 s. In our example the stimulation frequency was 11.55 Hz, the individual alpha rhythm 11 Hz.

The averaged MEG (Figure 6a, thick black line) indicates an increase of the alpha power after the flicker onset which is clearly represented in the GT spectrogram (Figure 6b) and CMWT scalogram (Figure 6c). The GT and the CMWT have similar time-frequency resolutions at the alpha rhythm. Both have the time resolution $\sigma_t = 0.1$ s (at 10 Hz in the CMWT, for all frequencies in the GT). The GT shows the 50-Hz-component (supply frequency) as a power line where the CMWT smears the component (lower frequency resolution) and additionally the higher time resolution produces a fine graining in time. The TFD shows a high frequency resolution of the MEG alpha activity (Figure 6d). An advantage of the WVD can be demonstrated by such an example, i.e. a mono-component signal whose frequency is a weakly varying time function. It should be underlined that we have chosen such exemplary signals to clearly show the similarities and differences of the methods and that in usual applications the WVD interference terms hamper a valid interpretation of the results which can be obtained around the alpha line. As mentioned above, the WVD does not provide instantaneous phase information. Therefore, the phase-locking analysis can only be carried out for GT and CMWT. We used the amplitude-independent phase-locking index (PLI) which indicates the locking and stability of the instantaneous phase to a time event [9], in our case to the flicker onset, over the trials. A phase-locking of the alpha rhythm

Figure 6 MEG spectrograms/scalogram/TFD (b-d) and phase-locking maps (e-f) analysis before and after a flicker stimulation. The flicker onset is at 0 s. An overlay of the 20 trials (MEG signals) are depicted in (a), the thick line in (a) shows the averaged course. The spectrograms of the GT (b), the CMWT’s scalogram (c) and the WVD-based TFD are shown. The corresponding GT’s and CMWT’s phase-locking maps are (e) and (f). The WVD does not provide phase information.
occurs after the flicker onset and is maintained during the flicker, i.e. the alpha is entrained to the flicker frequency. For analysis of the alpha activity the chosen time-frequency resolution seems to be optimal. That is different in the case of transient gamma oscillations at 40 Hz and 60 Hz, which can be observed in the GT’s PLI-map (Figure 6e). The CMWT shows a double power ridge configuration (Figure 6f) in the corresponding time-frequency regions indicating a possible occurrence of a double spindle as shown in Figure 5Ca. Perhaps on the one hand the time-resolution of the GT in the gamma range is too low, i.e. a blurring in time might occur, on the other hand the time-resolution of the CMWT seems too high and its corresponding frequency resolution is too low. An optimal time-frequency resolution cannot be provided because the real location of the oscillatory gamma activity in time and frequency is the objective of the analysis and unknown. However, according to our results a transient phase-locking after the flicker onset is most probably existent in the gamma frequency range. Additionally, the results may lead to the conclusion that the GT, ST, and CMWT seem to work best if the analysis wavelets are equal to the signal parts that are to be analyzed. So if one assumes that the signal consists of just one Gabor wavelet (e.g. gamma oscillation at 40 Hz), one can choose the envelope of this atom as a time window for the GT and thus achieve a perfect match between signal and analysis wavelet.

Remark 3.5.1-2

The results (measured MEG signals) of the GT-spectrogram and CMWT-scalogram analysis using different time-frequency resolutions are visualized by the videos tfrGab.avi and tfrMor.avi which are available as supplementary files.

Frequency-selective HT was also applied to these data [12]. The instantaneous phase and instantaneous frequency can only be reliably obtained after an optimal narrow-banded band pass filtering (Figure 9). For such a decision results of spectrogram/scalogram/TFD analyses can be used by which the frequency bands of interest can be identified. In combination with narrow-band filtering, the HT can be used for the amplitude (envelope) and frequency/phase decomposition of an extracted signal component, where the frequency is given by the center frequency of the pass band. An exemplary result of the HT application will be shown in comparison with the result of the HHT in Section 4.

3.5.2 Interference (Cross) Terms

The time-frequency resolutions of the GT/CMWT are closely linked with the extent of interference (cross) terms in the spectrogram/scalogram (see [10]). For the GT, ST, and CMWT it is valid that interference terms are restricted to regions in the time-frequency plane in which peaks overlap, i.e. interference terms are low if the distance between two oscillation frequencies is high or and the frequency resolution is sufficiently high (Figure 5Bd–Bc). However, ST and CMWT have frequency-dependent time-frequency resolutions, i.e. interference terms are more probable for higher frequencies because of the lower frequency resolution. In this respect GT may show better results than ST and CMWT.

Due to the quadratic nature of WVD, interference always occurs and can overlap in the time-frequency plane of signal components, i.e. it can hamper the interpretation of the WVD’s TFD. Interferences cannot be prevented [21], they are the price to be paid for good time- and frequency resolution, i.e. there is a trade-off between the extent of interference and the number of good properties. Interference geometry can be simply described for the two-component case and can be generalized for an arbitrary number of components [10]: Two signal components in the time-frequency plane provide a contribution to an interference term in such a way that each point of the interference term is located at the geometrical mid-point between the corresponding component points. Additionally, the interference term shows an oscillation with a frequency proportional to the distance of the contributing components (component points). The oscillation of the interference term occurs orthogonal to the line linking the components (component points) (Figure 5Bf). By using the analytical signal instead of the measured signal the interference terms can be reduced. In addition, the extent of interference terms can be reduced by smoothing, however smoothing causes a decreased time-frequency resolution. A two-dimensional smoothing (Gaussian smoothing functions) of the WVD (SPWVD) allows a continuous passage from the spectrogram to the WVD, i.e. the Heisenberg uncertainty principle cannot be undermined.

Remark 3.5.2

It was shown that time-frequency resolution plays an important role for a reliable analysis of transient oscillatory signal components whose frequency components can be distributed over the whole frequency range. In particular the properties of the WVD’s interference terms are troublesome for the interpretation of the TFD. To overcome these drawbacks, signal-adaptive approaches have been developed.

4. Signal-adaptive Techniques and their Applications

The simultaneous application of GT and CMWT to detect phase-locked gamma oscillations demonstrates that an optimal time-frequency resolution for the analysis of transient oscillations can only be achieved on the basis of a priori information. This requires some kind of preceding feature-driven decomposition of the signal that automatically generates a view of it that is suitable to its dynamics. Mallat and Zhang introduced in 1993 the Matching-Pursuit (MP) algorithm [11] that decomposes a signal into a sum of atoms from a given dictionary. Given that separation, a power distribution is generated in the time-frequency plane. So the first part of the algorithm is the decomposition of the signal into dictionary atoms.

For reasons of comparability to GT and CMWT, the MP algorithm which uses a Gabor atom dictionary D should be described and applied here. According to Equation 23, a real-valued Gabor atom consists only of a cosine term, i.e. in Figure 4 only the black signal is used.
Accordingly, $D$ is composed of cosine functions of (theoretically infinite) different frequencies ("modulation"-parameter $\omega$) which are windowed by a defined (theoretically infinitive) number of Gaussian time windows (\cite{eq. 25}), width parameter $\sigma_t$ and centered as shown in \cite{Figure 4} for each frequency at each window time $\tau$ (translation parameter $\tau$). Such a dictionary would be extremely redundant. Therefore, in practice the MP will be performed by a dictionary with a reduced number of atoms, i.e. for a defined sub-set of the parameters $\omega_{\tau}, \sigma_{t},$ and $\tau$. By an iterative algorithm a small number of atoms $g_n(t)\parallel g_n(t) = 1$ (iteration index $n$) and their weighting coefficients $a_n$ are determined which decompose the signal $x(t)$ into a weighted sum of Gabor atoms plus a residual

$$x(t) = \sum_{n=1}^{M} a_n \cdot g_n(t) + r_M. \quad \text{(46)}$$

The iteration algorithm runs as follows. The initial residual is set to be $r_0(t) = x(t)$ (zeroth approximation of $x(t)$). For the computation of the first approximation of the signal, the sum of products (inner product $= \text{scalar product}$) between $x(t)$ and each atom $g(t)$ of the dictionary is computed. In this way the atom which provides the highest inner product, i.e. the highest correlation (the inner product is the numerator of the linear correlation coefficient between both signals, i.e. their covariance), can be detected. This atom $g_1(t)$, multiplied by value of the inner product (weighting coefficient $a_1$), will be subtracted from the signal which yields to the first residual $r_1(t)$

$$r_1(t) = x(t) - \langle x(t), g_1(t) \rangle \cdot g_1(t) =$$

$$\text{Remark 4-1}\quad r_1(t) = \langle r_1(t), g_2(t) \rangle \cdot g_1(t).$$

The notation $\langle \ldots, \ldots \rangle$ is used for the inner product between two signals. In the 2nd iteration step, the residual $r_1(t)$ is used "as signal" and the atom $g_2(t)$ with the highest correlation with $r_1(t)$ is computed and subtracted, so that the second residual $r_2(t)$ results. In this way this "greedy" procedure continues, so that the residual for the $n$-th iteration step can be generalized by

$$r_n(t) = r_{n-1}(t) - \langle r_{n-1}(t), g_n(t) \rangle \cdot g_n(t). \quad \text{(48)}$$

This recurs until a stopping criterion is met, e.g. a pre-defined percentage of approximated energy. Thereafter, the weighted Gabor atoms can be analyzed separately.

Due to the fact that the Gabor atoms are mono-component signals per definition, the application of the WVD does not provide interference terms and has an excellent time-frequency resolution.

In \cite{Figure 7} the results of a WVD and MP analysis near the flicker onset (0 s) are represented. We have focused on results which demonstrate the problematic nature of the interference caused by WVD and its elimination by the MP approach.

The increase in alpha amplitude and frequency (towards the flicker frequency) after flicker onset can be shown by both TFDs (\cite{Figures 7b and 7c}). These physiological effects are known as entrainment effects. The WVD-typical interference terms can be observed near the alpha oscillation and they occur strongly at low frequencies. In the MP-TFD (\cite{Figure 7c}) no interference terms can be observed. The entrainment-dependent time-frequency evolution of the alpha can be quantified with optimal time-frequency resolution.

The MP(-WVD) approach provides an excellent time-frequency resolution without interference terms because the atoms analyzed are mono-component signals per definition. However, one key problem remains. The WVD provides no phase information. It is a power-only distribution. For phase analyses, this should be an exclusion criterion. If the MP decomposition uses a Gabor atom dictionary, then the problems of the GT, ST, and CMWT with regard to the selection of an appropriate time-frequency resolution can be solved. In 2011 Wacker and Witte \cite{12} introduced a combination of the MP and the GT algorithm (matched Gabor transform MGT) for the optimization of time-variant spectrogram analysis and for adaptive phase extraction.

The MGT adopts the decomposition of the MP approach and combines it with the finding that analysis methods seem to work best if the analysis wavelets match the signal parts which are to be analyzed. We assume that the signal is composed of Gabor atoms only, then a MP with a Gabor dictionary is optimal to analyze each real-
valued atom $g_n(t)$ with its own analysis time-window, determined by $\sigma_n$, to generate a set of complex time-frequency planes \( \{ y_n(t, \omega) \}_{n=1}^{M} \) and finally combine the weighted \( (a_n) \) planes to obtain a time-frequency plane \( y(t, \omega) \):
\[
y(t, \omega) = \sum_{n=1}^{M} a_n \cdot y_n(t, \omega)
\]  
(49)
with (according to Eq. 30)
\[
y_n(t, \omega) = g_n(t) \ast g(t, \omega, \sigma_n).
\]  
(50)

Remark 4-2

Firstly, the resulting time-frequency plane \( y(t, \omega) \) has multiple intrinsic time-frequency resolutions and cannot be obtained by a single GT. By means of MGT each transient signal component is analyzed with an optimally fitted time-frequency resolution that is determined by the signal. In contrast to the manually adaptable GT and CMWT, the MGT provides an objective result which can be used to characterize the natural appearance of transient components. Secondly, the atom-related complex time-frequency plane \( y_n(t, \omega) \) is the analytic signal of the atom and therefore the instantaneous phase properties of the signal can be reconstructed.

Therefore, it would be interesting to compare the phase-locking results between GT and MGT because we have seen that GT and CMWT provide different results and thereby lead to different interpretations. To give a simple example, we want to analyze the phase-locking effect in evoked brain activity data. Having multiple repetitions of the experiment, phase-locking means that the extracted phase values in a certain region of the time-frequency plane show only a small jitter among the repetitions. This effect can be measured by the phase-locking index (PLI, see [9] for details). In Figure 8 the comparison between GT- and MGT-based PLI analysis is performed.

PLI characteristics of the onset-related alpha and the gamma oscillation (40 Hz) are of particular interest. The MGT’s optimized time-frequency-resolution results in a better separability and thereby in a more reliable interpretation of phase-locked (evoked) activity. Immediately with the flicker onset (at 0 s) the phase-locking of the alpha rhythm begins and within the first 200 ms after burst onset a transient phase-locked beta oscillation (16 Hz) can be clearly distinguished. The GT-based PLI does not allow such an interpretation. For the transient phase-locked gamma oscillation (at 40 Hz) the GT-based PLI has a duration of approximately 1 s and the MGT-based one shows a splitting up of the PLI course. In contrast to the GT-based PLI the MGT’s time-frequency-resolution is adapted to the transient nature of the gamma activity. The comparison between non-adaptive (GT, CMWT) and adaptive (MGT) PLI approaches shows the superiority of the MGT. However, the most important advantage of the MGT is that the optimal time-frequency-resolution can be obtained automatically and for all frequencies.

The empirical mode decomposition (EMD), introduced by Huang in 1998 [13], has become increasingly established in biomedicine in the last few years (for an overview see [22]). The combination of EMD with HT is known as the Hilbert-Huang Transform (HHT) and enables a signal-adaptive decomposition and demodulation of the extracted intrinsic signal components (intrinsic mode functions IMFs). The IMFs are mono-component signals, i.e. it can be assumed that HT is the appropriate method for their analysis. The representation of the envelope and the instantaneous frequency of the IMFs (Eqs. 12 and 14) is called the Hilbert spectrum.

The EMD is a decomposition of the form
\[
x(t) = \sum_{n=1}^{M} c_n(t) + r_M(t),
\]  
(51)
where \( c_n(t) \) is the \( n \)-th IMF and \( r_M(t) \) is the residuum after the extraction of \( M \) components. Each IMF has the following characteristics:
- The number of its local extrema and the number of zero-crossings differs at most by one.
- At any point in time, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is zero.

The EMD to obtain such IMFs can be formulated as in [23]:

![Figure 8](image_url)

**Figure 8** PLI analysis of 20 MEG trials (a) before and after flicker onset (0 s). GT-related (b) and MGT-related (c) PLI-maps indicate a phase-stabilization of the alpha oscillations during the flicker stimulation.
1. Given the original signal \( x(t) \), set \( r_0(t) = x(t), n = 1 \).
2. Extract the \( n \)-th IMF using the sifting procedure:
   a) Set \( j = 1 \) and \( h_{j-1}(t) = r_{n-1}(t) \).
   b) Identify the successive local minima and the local maxima for \( h_{j-1}(t) \).
   c) Interpolate the local minima and the local maxima with cubic splines to form an upper \( e_{\text{max}}^{j-1}(t) \) and lower \( e_{\text{min}}^{j-1}(t) \) envelope.
   d) Compute the instantaneous mean of the envelopes
      \[
      m_{j-1}(t) = \left( e_{\text{max}}^{j-1}(t) + e_{\text{min}}^{j-1}(t) \right) / 2
      \]  
      (52)
      and determine a new estimate
      \[
      h_{j}(t) = h_{j-1}(t) - m_{j-1}(t),
      \]  
      (53)
      such that
      \[
      e_{\text{min}}^{j-1}(t) \leq h_{j}(t) \leq e_{\text{max}}^{j-1}(t) \text{ for all } t
      \]  
      (54)
      Set \( j = j + 1 \).
3. Obtain an improved residuum
      \[
      r_{n}(t) = r_{n-1} - c_{n}(t).
      \]  
      (55)
   e) Repeat steps 2b – 2d until \( h_{j}(t) \) satisfies the condition of an IMF.
4. Set \( n = n + 1 \). Repeat step 2 until the number of extrema in \( r_{n}(t) \) is less than 2.

The iterative EMD algorithm produces IMFs which are in line with this definition, i.e. non-stationary and e.g. periodic (non-sinusoidal) signal components which are characterized by higher harmonics can be extracted. This was used to promote the HHT, but it contradicts the application of HT because higher harmonics spread over a wide frequency range. Further problems are incurred with the use of EMD (see Discussion).

In Figure 9a the HT-based instantaneous frequency computation of the band-pass filtered alpha-oscillation and of the IMF which encompasses the oscillatory alpha activity is shown. The band-pass has an optimal center frequency because it is identical with the flicker frequency (11.55 Hz) and the alpha frequency becomes entrained to the flicker-frequency during stimulation. The narrow-banded alpha oscillation shows a smooth instantaneous frequency which becomes almost constant 500 ms after the flicker onset and remains stable during flickering. This is a physiological, frequency-related effect of the entrainment which is accompanied by a stabilization of phase-locking (Figure 8b–c). The alpha-related IMF is characterized by a distorted instantaneous frequency, where the distortions are reduced during flicker stimulation, i.e. it is very difficult to give a correct interpretation. Nevertheless, its mean course is similar to the course of the instantaneous frequency of the narrow-banded alpha component. The reason for the distortion is the EMD’s limited band-pass filter effect. The EMD of a Gaussian noise process can be compared with a dyadic band-pass filter bank, i.e. the distance of center frequencies of the IMF’s frequency bands doubles from IMF to IMF.

5. Discussion

The purpose of this paper is to show that for a reliable application of time-frequency techniques knowledge with regard to their mathematical fundamentals, the similarities as well as differences between the approaches and their drawbacks, is required. Only the appropriate selection of the method and of its parameter settings will ensure the readability of the TFRs and reliability of results. Unfortunately, an appropriate selection essentially depends on the (unknown) signal characteristics which should be analyzed, i.e. it is beneficial to have a priori information with regard to the signal characteristics. Only when a signal’s time-frequency characteristics strongly correspond with the frequency-(in)dependent time-frequency resolution of the analysis method, can the application be considered optimal. This simple fact is helpful by unveiling some “myths” with regard to the superiority or optimality of a specific method, as the above given assumption does not correspond to a ‘real’ situation. Accordingly, it is very difficult to give general recommendations for selection of a particular method and relevant
parameter settings (see remark 3.5-1). However, one can learn by examples. Therefore, we complemented the mathematical results by analysis derived from signal simulations and by using MEG signals with specific properties, to illuminate the advantages and disadvantages of particular methods. This should allow some important cornerstones of parameter settings; GT, ST, and CMWT provide identical results only for one frequency component and only if the time-frequency resolutions are identical. Otherwise the specific characteristics of time-frequency resolutions provide different views of the signal’s dynamics. As an example, for the CMWT the frequency of the mother wavelet can be adapted to the frequency of the component of interest. In our definition of the Morlet wavelet (Eq. 33), the selection of \( \omega_a \) determines the time-frequency resolution.

An additional control term can be introduced to tune the time-frequency resolution for the mother wavelet, i.e. the duration of the mother wavelet at \( \omega_a \) can be determined by changing the width of the Gaussian window. This allows use of a matched mother wavelet which is similar to the expected transient oscillation in the signal. Jones and Parks [20] used the term “matched windowing”.

A good time-frequency resolution of the WVD has to be bought with the price of strong interference terms which could limit a valid interpretation of the TFDs. The WVD is perfect for analyzing (academic) signals whose frequency is a linear function of time (linear chirp signals). For such a class of signals the power ridge (line) is perfectly localized on the signal’s instantaneous frequency, i.e. optimally localized in the frequency domain. As shown by our simulation in Figure 5, WVD is also superior in the time domain for the Dirac impulse.

The complexity of problems with regard to the time-frequency resolution and interference terms becomes perpetuated when the time-frequency methodologies are used as a basis for further computations, e.g. the time-variant cross-spectrum (or coherence) for connectivity analyses and time-variant bi-spectral measures (quadratic phase couplings QPC) for synchronization analysis (e.g. [24]). This can be exemplified by a time-variant 3rd-order TFD for QPC analysis [25]. The advantage of the TFD that time and frequency resolution can be adjusted independently can be used to detect amplitude modulations which are a source of QPC in several transient EEG patterns. QPC occurs when the triple of spectral components at \( \omega_1, \omega_2 \) and \( \omega_1 + \omega_2 \) (or \( \omega_2 - \omega_1 \)) \((\omega_2 > \omega_1)\) exists and the phases of the components are in the same relationship, i.e. \( \varphi_1(t), \varphi_2(t), \) and \( \varphi(t) + \varphi(0) \) (or \( \varphi(t) + \varphi(0) \)). It is obvious that to distinguish between the components a high frequency resolution is required. The QPC patterns occur transiently [9, 25], i.e. a high time resolution is also required.

Both the selection of an optimal time-frequency resolution for each frequency and the reduction of interference terms in the spectrogram/scalogram/TFD can be automatically solved by using the MP approach by which the signal is decomposed into atoms and which then uses the sum of atom representations as the representation of the whole signal. For the reduction of interference terms alternative approaches have been proposed, e.g. modification of the spectrograms/scalograms/TFDs by image processing or reassignment methods [26]. A combination of MP and GT was recently introduced by Wacker and Witte [12] for adaptive phase extraction. The MGT provides optimal time-frequency resolutions for all signal components if they are of the Gabor wavelet class. Therefore, for many biomedical phase-locking and synchronization analyses GT, ST, and CMWT can be beneficially replaced by MGT. However, MP can have a bias according to its dictionary. So if the atoms from the dictionary are shifted on a discrete time grid to cover the whole time range, the discrete positions automatically introduce a bias. To compensate for that, stochastic dictionaries have been introduced [27], where decomposition is carried out for many different, randomly perturbed dictionaries. Averaging of the energy distributions generates the final result. Another drawback of the MP concept is that one has to accept “sub-optimal” solutions which are generated by the “greedy” strategy as shown in Section 4 (Eqs. 47, 48).

The EMD decomposes time-frequency signals into zero-mean amplitude modulation and frequency modulation components (Eq. 17), the so-called intrinsic mode functions (IMFs). According to the calculation procedure, IMFs are non-stationary, mono-component signals. The HT is a linear transform and not appropriate to analyze, e.g. periodic signals which are composed of higher harmonics. Therefore, it is surprising that the combination of EMD and HT, the HHT, has been proposed as a time-variant, non-linear analysis method. Additionally, the EMD has its own method-intrinsic drawbacks, e.g. EMD’s filter characteristics are not always sufficient for biomedical applications and an artefactual mode-mixing occurs. Mode-mixing means that an IMF can include portions of other IMFs, i.e. of other frequency scales [28]. Numerous modifications of EMD have been proposed to avoid or to reduce mode mixing and alternative decomposition methods have been developed to replace EMD (e.g. [29]).

The EMD can be seen as a dyadic filter bank (for Gaussian noise) [30], i.e. the center frequencies of the pass bands are doubled from band to band. For several biomedical applications such filter characteristics are not sufficient and therefore HT provides non-interpretable results, e.g. due to the fact that higher harmonics of a non-linear oscillation are not sufficiently rejected by the filter effect. That is the reason why signal-adaptive HT approaches have been introduced which use adaptive narrow-banded band pass filters for the selection of biomedical oscillatory signal components [31]. For analysis methods other than the HT, EMD filter characteristics are turned into an advantage. It is known that filtering leads to wrong results in connectivity analysis [32] by using multivariate autoregressive models. In such applications band pass filters can be replaced by the EMD [33].

This review outlines the methodological fundamentals of the most frequently used time-variant non-parametric time-frequency techniques, their properties and the corresponding consequences for their application. Several toolboxes and corresponding tutorials are available to support and to facilitate their implementation,
application, and interpretation of results. However, recent mathematical developments for biomedical applications have progressed already far beyond these approaches. For example, modified versions of the ST has been recently introduced [34], the MP has been extended to spatio-temporal (multichannel) EEG/MEG analyses [35], and the EMD is now utilized for multichannel [36] as well as multivariate approaches [37]. Thus reviews are an ongoing necessity to periodically assess state-of-the-art knowledge and know-how, with the aim of aiding in the choice and optimal use of the wide variety of methods to understand biomedical signals within both biomedical and clinical research, ultimately to the benefit of patients.

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